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Inequalities for Stieltjes integrals with convex integrators and applications

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Abstract

Inequalities for a Grüss type functional in terms of Stieltjes integrals with convex integrators are given. Applications to the Čebyšev functional are also provided.

Keywords: Stieltjes integral; Grüss inequality; Čebyšev inequality; Convex functions

1. Introduction

In [3], the authors have considered the following functional:

$$D(f;u) := \int_a^b f(x)\mathrm{d}u(x) - [u(b)-u(a)] \cdot \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t, \tag{1.1}$$

provided that the Stieltjes integral $\int_a^b f(x)\mathrm{d}u(x)$ and the Riemann integral $\int_a^b f(t)\mathrm{d}t$ exist.

In [3], the following result in estimating the above functional has been obtained:

Theorem 1. *Let $f, u : [a,b] \to \mathbb{R}$ be such that u is Lipschitzian on $[a,b]$, i.e.,*

$$|u(x) - u(y)| \le L|x-y| \quad \textit{for any } x, y \in [a,b] \quad (L > 0) \tag{1.2}$$

and f is Riemann integrable on $[a,b]$.

If $m, M \in \mathbb{R}$ are such that

$$m \le f(x) \le M \quad \textit{for any } x \in [a,b], \tag{1.3}$$

then we have the inequality

$$|D(f;u)| \le \frac{1}{2}L(M-m)(b-a). \tag{1.4}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

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In [2], the following result complementing the above has been obtained:

Theorem 2. *Let $f, u : [a, b] \to \mathbb{R}$ be such that u is of bounded variation on $[a, b]$ and f is Lipschitzian with the constant $K > 0$. Then we have*

$$|D(f; u)| \leq \frac{1}{2} K(b-a) \bigvee_a^b (u). \tag{1.5}$$

The constant $\frac{1}{2}$ is sharp in the above sense.

For a function $u : [a, b] \to \mathbb{R}$, define the associated functions Φ, Γ and Δ by:

$$\Phi(t) := \frac{(t-a)u(b) + (b-t)u(a)}{b-a} - u(t), \quad t \in [a, b]; \tag{1.6}$$
$$\Gamma(t) := (t-a)[u(b) - u(t)] - (b-t)[u(t) - u(a)], \quad t \in [a, b]$$

and

$$\Delta(t) := \frac{u(b) - u(t)}{b-t} - \frac{u(t) - u(a)}{t-a}, \quad t \in (a, b).$$

In [1], the following subsequent bounds for the functional $D(f; u)$ have been pointed out:

Theorem 3. *Let $f, u : [a, b] \to \mathbb{R}$.*

(i) *If f is of bounded variation and u is continuous on $[a, b]$, then*

$$|D(f; u)| \leq \begin{cases} \sup_{t \in [a,b]} |\Phi(t)| \bigvee_a^b (f), \\ \frac{1}{b-a} \sup_{t \in [a,b]} |\Gamma(t)| \bigvee_a^b (f), \\ \frac{1}{b-a} \sup_{t \in (a,b)} [(t-a)(b-t)|\Delta(t)|] \bigvee_a^b (f). \end{cases} \tag{1.7}$$

(ii) *If f is L-Lipschitzian and u is Riemann integrable on $[a, b]$, then*

$$|D(f; u)| \leq \begin{cases} L \int_a^b |\Phi(t)| \mathrm{d}t, \\ \frac{L}{b-a} \int_a^b |\Gamma(t)| \mathrm{d}t, \\ \frac{L}{b-a} \int_a^b (t-a)(b-t)|\Delta(t)| \mathrm{d}t. \end{cases} \tag{1.8}$$

(iii) *If f is monotonic nondecreasing on $[a, b]$ and u is continuous on $[a, b]$, then*

$$|D(f; u)| \leq \begin{cases} \int_a^b |\Phi(t)| \mathrm{d}f(t), \\ \frac{1}{b-a} \int_a^b |\Gamma(t)| \mathrm{d}f(t), \\ \frac{1}{b-a} \int_a^b (t-a)(b-t)|\Delta(t)| \mathrm{d}f(t). \end{cases} \tag{1.9}$$

The case of monotonic integrators is incorporated in the following two theorems [1]:

Theorem 4. *Let $f, u : [a, b] \to \mathbb{R}$ be such that f is L-Lipschitzian on $[a, b]$ and u is monotonic nondecreasing on $[a, b]$, then*

$$\begin{aligned}|D(f; u)| &\le \frac{1}{2}L(b-a)[u(b)-u(a)-K(u)] \\ &\le \frac{1}{2}L(b-a)[u(b)-u(a)],\end{aligned} \tag{1.10}$$

where

$$K(u) := \frac{4}{(b-a)^2}\int_a^b u(x)\left(x-\frac{a+b}{2}\right)\mathrm{d}x \ge 0. \tag{1.11}$$

The constant $\frac{1}{2}$ in both inequalities is sharp.

Theorem 5. *Let $f, u : [a, b] \to \mathbb{R}$ be such that u is monotonic nondecreasing on $[a, b]$, f is of bounded variation on $[a, b]$ and the Stieltjes integral $\int_a^b f(x)\mathrm{d}u(x)$ exists. Then*

$$\begin{aligned}|D(f; u)| &\le [u(b)-u(a)-Q(u)]\bigvee_a^b(f) \\ &\le [u(b)-u(a)]\bigvee_a^b(f),\end{aligned} \tag{1.12}$$

where

$$Q(u) := \frac{1}{b-a}\int_a^b \operatorname{sgn}\left(x-\frac{a+b}{2}\right)u(x)\mathrm{d}x \ge 0. \tag{1.13}$$

The first inequality in (1.12) is sharp.

The main aim of this work is to establish new sharp inequalities for the functional $D(\cdot;\cdot)$ on the assumption that the integrator u in the Stieltjes integral $\int_a^b f(x)\mathrm{d}u(x)$ is convex on $[a, b]$. Applications for the Čebyšev functional of two Lebesgue integrable function are also given.

2. Inequalities for convex integrators

The following result may be stated:

Theorem 6. *Let $u : [a, b] \to \mathbb{R}$ be a convex function on $[a, b]$ and $f : [a, b] \to \mathbb{R}$ a monotonic nondecreasing function on $[a, b]$. Then*

$$\begin{aligned}0 &\le D(f; u) \\ &\le 2\cdot\frac{u'_-(b)-u'_+(a)}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)f(t)\mathrm{d}t \\ &\le \begin{cases}\frac{1}{2}[u'_-(b)-u'_+(a)]\max\{|f(a)|,|f(b)|\}(b-a); \\ \frac{1}{(q+1)^{\frac{1}{q}}}[u'_-(b)-u'_+(a)]\,\|f\|_p\,(b-a)^{\frac{1}{q}} \\ \qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ [u'_-(b)-u'_+(a)]\,\|f\|_1.\end{cases}\end{aligned} \tag{2.1}$$

The constants 2 and $\frac{1}{2}$ are best possible.

Proof. Integrating by parts in the Stieltjes integral, we have

$$\begin{aligned}\int_a^b \Phi(t)\mathrm{d}f(t) &= \left[\frac{(t-a)u(b)+(b-t)u(a)}{b-a} - u(t)\right] f(t)\Big|_a^b \\ &\quad - \int_a^b f(t)\mathrm{d}\left[\frac{(t-a)u(b)+(b-t)u(a)}{b-a} - u(t)\right] \\ &= [u(b)-u(b)]f(b) - [u(a)-u(a)]f(a) - \int_a^b f(t)\left[\frac{u(b)-u(a)}{b-a}\mathrm{d}t - \mathrm{d}u(t)\right] \\ &= \int_a^b f(t)\mathrm{d}u(t) - \frac{u(b)-u(a)}{b-a}\int_a^b f(t)\mathrm{d}t = D(f;u), \end{aligned} \tag{2.2}$$

for any u a continuous function on $[a, b]$ and f of bounded variation on $[a, b]$.

This identity has been established in [1]. In equation (56) in [1], there is a typographical error in the first equation. The definition of Φ is provided in (1.6).

The fact that $D(f; u) \geq 0$ for u convex and f monotonic nondecreasing on $[a, b]$ has been proven earlier in [1]. For the sake of completeness we give here a different and simpler proof as well.

Since u is convex, then

$$\begin{aligned}\frac{t-a}{b-a}\cdot u(b) + \frac{b-t}{b-a}\cdot u(a) &\geq u\left[\frac{(t-a)b+(b-t)a}{b-a}\right] \\ &= u(t),\end{aligned}$$

for any $t \in [a, b]$. Thus, $\Phi(t) \geq 0$ for $t \in [a, b]$ and since f is monotonic nondecreasing, then $\int_a^b \Phi(t)\mathrm{d}f(t) \geq 0$.

Now, for any convex function $\Phi : [a, b] \to \mathbb{R}$ we have

$$\Phi(x) - \Phi(y) \geq \Phi'_{\pm}(y)(x-y) \quad \text{for any } x, y \in (a, b) \tag{2.3}$$

where $\Phi'_{\pm}$ are the lateral derivatives of the convex function Φ. Then, on using (2.3), we have

$$u'(t) - u(b) \geq u'_{-}(b)(t-b).$$

If we multiply this inequality by $t - a \geq 0$, we get

$$(t-a)u(t) - (t-a)u(b) \geq u'_{-}(b)(t-b)(t-a). \tag{2.4}$$

Similarly, we have

$$(b-t)u(t) - (b-t)u(a) \geq u'_{+}(a)(t-a)(b-t). \tag{2.5}$$

Adding (2.4) with (2.5) and dividing by $b - a$, we deduce:

$$u(t) - \frac{(t-a)u(b)+(b-t)u(a)}{b-a} \geq \frac{(b-t)(t-a)}{b-a}[u'_{+}(a) - u'_{-}(b)]$$

giving the inequality:

$$0 \leq \frac{(t-a)u(b)+(b-t)u(a)}{b-a} - u(t) \leq \frac{(b-t)(t-a)}{b-a}[u'_{-}(b) - u'_{+}(a)]. \tag{2.6}$$

Integrating this inequality, we get

$$\int_a^b \Phi(t)\mathrm{d}f(t) \leq \frac{[u'_{-}(b) - u'_{+}(a)]}{b-a}\int_a^b (b-t)(t-a)\mathrm{d}f(t).$$

On the other hand

$$\begin{aligned}\int_a^b (b-t)(t-a)\mathrm{d}f(t) &= f(t)(b-t)(t-a)\Big|_a^b - \int_a^b f(t)[-2t+(a+b)]\mathrm{d}t \\ &= 2\int_a^b f(t)\left(t - \frac{a+b}{2}\right)\mathrm{d}t,\end{aligned}$$

giving the second inequality in (2.1).

Utilising Hölder's inequality, we have

$$\int_a^b \left(t - \frac{a+b}{2}\right) f(t)\mathrm{d}t \le \begin{cases} \sup\limits_{t\in[a,b]} |f(t)| \displaystyle\int_a^b \left|t - \frac{a+b}{2}\right| \mathrm{d}t; \\ \left(\displaystyle\int_a^b |f(t)|^p \mathrm{d}t\right)^{\frac{1}{p}} \left(\displaystyle\int_a^b \left|t - \frac{a+b}{2}\right|^q \mathrm{d}t\right)^{\frac{1}{q}} \\ \qquad \text{if } p > 1,\ \dfrac{1}{p} + \dfrac{1}{q} = 1; \\ \sup\limits_{t\in[a,b]} \left|t - \dfrac{a+b}{2}\right| \displaystyle\int_a^b |f(t)|\mathrm{d}t, \end{cases}$$

$$= \begin{cases} \dfrac{1}{4} \max\{|f(a)|, |f(b)|\} (b-a)^2; \\ \dfrac{1}{2} \cdot \dfrac{1}{(q+1)^{\frac{1}{q}}} \|f\|_p (b-a)^{1+\frac{1}{q}} \\ \qquad \text{if } p > 1,\ \dfrac{1}{p} + \dfrac{1}{q} = 1; \\ \dfrac{1}{2} \|f\|_1 (b-a), \end{cases}$$

and the last part of (2.1) is proved.

Now, for the best possible constant.

Assume that (2.1) holds with a constant C instead of 2, i.e.,

$$D(f; u) \le C \cdot \frac{u'_-(b) - u'_+(a)}{b-a} \int_a^b \left(t - \frac{a+b}{2}\right) f(t)\mathrm{d}t, \tag{2.7}$$

where u is convex on $[a, b]$ and f is monotonic nondecreasing on $[a, b]$.

Consider $u(t) := |t - \frac{a+b}{2}|$ and $f(t) = \operatorname{sgn}(t - \frac{a+b}{2})$. Then u is convex on $[a, b]$ and f is monotonic nondecreasing on $[a, b]$. We have

$$\begin{aligned} D(f; u) &= \int_a^{\frac{a+b}{2}} (-1)\mathrm{d}\left(\frac{a+b}{2} - t\right) + \int_{\frac{a+b}{2}}^b (+1)\mathrm{d}\left(t - \frac{a+b}{2}\right) \\ &= \int_a^b \mathrm{d}t = (b-a), \\ u'_-(b) - u'_+(a) &= 2 \end{aligned}$$

and

$$\begin{aligned} \int_a^b \left(t - \frac{a+b}{2}\right) f(t)\mathrm{d}t &= \int_a^b \left(t - \frac{a+b}{2}\right) \operatorname{sgn}\left(t - \frac{a+b}{2}\right) \mathrm{d}t \\ &= \int_a^b \left|t - \frac{a+b}{2}\right| \mathrm{d}t = \frac{(b-a)^2}{4}. \end{aligned}$$

Therefore, from (2.7) we get

$$b - a \le \frac{C(b-a)}{2},$$

giving that $C \ge 2$. The fact that $\frac{1}{2}$ is best possible goes likewise and we omit the details. □

The following result may be stated as well:

Theorem 7. *Let $u : [a, b] \to \mathbb{R}$ be a continuous convex function on $[a, b]$ and $f : [a, b] \to \mathbb{R}$ a function of bounded variation on $[a, b]$. Then*

$$|D(f; u)| \le \frac{1}{4}[u'_-(b) - u'_+(a)](b - a) \bigvee_a^b (f), \tag{2.8}$$

where $\bigvee_a^b (f)$ denotes the total variation of f on $[a, b]$.

The constant $\frac{1}{4}$ is best possible in (2.8).

Proof. It is well known that if $p : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and $v : [a, b] \to \mathbb{R}$ is of bounded variation on $[a, b]$, then the Stieltjes integral $\int_a^b p(t)\mathrm{d}v(t)$ exists and

$$\left| \int_a^b p(t)\mathrm{d}v(t) \right| \le \sup_{t\in[a,b]} |p(t)| \bigvee_a^b (f). \tag{2.9}$$

Utilising the inequality (2.6) we have

$$\begin{aligned} \sup_{t\in[a,b]} \left| \frac{(t-a)u(b) + (b-t)u(a)}{b-a} - u(t) \right| &\le \frac{u'_-(b) - u'_+(a)}{b-a} \sup_{t\in[a,b]} [(b-t)(t-a)] \\ &= \frac{1}{4}(b-a)[u'_-(b) - u'_+(a)]. \end{aligned}$$

Now, utilising the identity (2.2) and the property (2.9), we have

$$\begin{aligned} |D(f; u)| &\le \sup_{t\in[a,b]} |\Phi(t)| \bigvee_a^b (f) \\ &\le \frac{1}{4}(b-a)[u'_-(b) - u'_+(a)] \end{aligned}$$

and the inequality (2.8) is proved.

Now, for the best constant.

Assume that there exists $D > 0$ such that

$$|D(f; u)| \le D[u'_-(b) - u'_+(a)](b - a) \bigvee_a^b (f) \tag{2.10}$$

provided that u is continuous convex and f is of bounded variation on $[a, b]$.

If we choose $u(t) = |t - \frac{a+b}{2}|$ and $f(t) = \operatorname{sgn}(t - \frac{a+b}{2})$, then (see the proof of Theorem 6)

$$D(f; u) = b - a, \quad u'_-(b) - u'_+(a) = 2 \quad \text{and} \quad \bigvee_a^b (f) = 2$$

giving in (2.10) that $b - a \le 4D(b - a)$ which implies $D \ge \frac{1}{4}$. □

The following result may be stated.

Theorem 8. *Let $u : [a, b] \to \mathbb{R}$ be a convex function on $[a, b]$ and $f : [a, b] \to \mathbb{R}$ a Lipschitzian function with the constant $L > 0$, i.e.,*

$$|f(t) - f(s)| \le L|t - s| \quad \text{for each } t, s \in [a, b]. \tag{2.11}$$

Then

$$|D(f; u)| \le \frac{1}{6}L(b - a)^2[u'_-(b) - u'_+(a)]. \tag{2.12}$$

Proof. It is well known that if $p : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ and $v : [a, b] \to \mathbb{R}$ is Lipschitzian with the constant $L > 0$, then the Stieltjes integral $\int_a^b p(t)\mathrm{d}u(t)$ exists and

$$\left|\int_a^b p(t)\mathrm{d}v(t)\right| \le L\int_a^b |p(t)|\mathrm{d}t. \tag{2.13}$$

Utilising the identity (2.6) and the property (2.13), we have

$$\begin{aligned}|D(f;u)| &\le L\int_a^b \left|\frac{(b-t)(t-a)[u'_-(b)-u'_+(a)]}{b-a}\right|\mathrm{d}t \\ &= \frac{L}{b-a}[u'_-(b)-u'_+(a)]\int_a^b (b-t)(t-a)\mathrm{d}t \\ &= \frac{1}{6}L(b-a)^2[u'_-(b)-u'_+(a)],\end{aligned}$$

and the theorem is proved. □

Remark 1. It is an open problem if the constant $\frac{1}{6}$ above is sharp.

3. Applications for the Čebyšev functional

For the Lebesgue integrable functions $f, g : [a, b] \to \mathbb{R}$ with fg an integrable function, consider the *Čebyšev functional* C, defined by

$$C(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \cdot \frac{1}{b-a}\int_a^b g(x)\mathrm{d}x.$$

The following result may be stated.

Proposition 1. *If f, g are monotonic nondecreasing functions, then*

$$\begin{aligned}0 &\le C(f,g) \\ &\le 2\cdot\frac{g(b)-g(a)}{b-a}\cdot\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)f(t)\mathrm{d}t \\ &\le \begin{cases}\frac{1}{2}[g(b)-g(a)]\max\{|f(a)|,|f(b)|\}; \\ \frac{1}{(q+1)^{\frac{1}{q}}}[g(b)-g(a)]\,\|f\|_p\,(b-a)^{\frac{1}{q}-1} \\ \qquad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{g(b)-g(a)}{b-a}\|f\|_1.\end{cases}\end{aligned} \tag{3.1}$$

The constants 2 *and* $\frac{1}{2}$ *are best possible.*

The proof is obvious by Theorem 6 on choosing $u : [a, b] \to \mathbb{R}$, $u(t) := \int_a^t g(s)\mathrm{d}s$. The sharpness of the constant follows as in the proof of Theorem 6 for $f, g : [a, b] = 1$, $f(t) = g(t) = \mathrm{sgn}(t - \frac{a+b}{2})$.

The following result may be stated as well:

Proposition 2. *If g is monotonic nondecreasing on $[a, b]$ and f is of bounded variation on $[a, b]$, then*

$$|C(f,g)| \le \frac{1}{4}[g(b)-g(a)]\bigvee_a^b(f). \tag{3.2}$$

The constant $\frac{1}{4}$ is best possible in (3.2).

The proof follows by Theorem 7 and the details are omitted.

Finally, on utilising Theorem 8, we can state

Proposition 3. *If g is monotonic nondecreasing and f is L-Lipschitzian on $[a, b]$, then*

$$|C(f, g)| \leq \frac{1}{6} L(b-a)[g(b)-g(a)].$$

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